Talk 2 - Gradient Ricci Soliton equation and consequences.

The GRS equation is

$$R_{ij} + \nabla_i \nabla_j f = \frac{\lambda}{2} g_{ij}. \qquad \text{--- } (1)$$

$\lambda > 0$ shrinking, $\lambda = 0$ steady, $\lambda < 0$ expanding.

f is called the potential function.

We expect relations b/w geometry of g and properties of f.

Lemma :- Suppose (M^n, g, λ, f) is a GRS and $(M^n, g) \underset{\text{iso}}{\cong} (M_1^{n_1}, g_1) \times (M_2^{n_2}, g_2)$. Then for any $x \in M_2^{n_2}$, $(M_1^{n_1}, g_1, f_1, \lambda)$ is a GRS w/ $f_1 : M_1^{n_1} \to \mathbb{R}$ is the restriction of f to $M_1^{n_1} \times \{x_2\} \cong M_1^{n_1}$.

i.e., if a GRS is a metric product then it is a product of GRSs.

proof:- $\because$ $g = g_1 + g_2$, if $X, Y \in \Gamma(TM_1) \cong \Gamma(T(M_1^{n_1} \times \{x_2\})) \subseteq \Gamma(TM)$

$$\begin{aligned}
(\nabla_g^2 f)(X, Y) &= X(Yf) - \langle \nabla_X^g Y, \nabla f \rangle_g \\
&= X(Yf) - \langle \nabla_X^{g_1} Y, \nabla f_1 \rangle_{g_1} \\
&= (\nabla_{g_1}^2 f_1)(X, Y)
\end{aligned}$$

($\nabla_X^g Y = \nabla_X^{g_1} Y$ is tangential to $M_1^{n_1} \times \{x_2\}$).

$\therefore$ $Ric_{g_1} + \nabla_{g_1}^2 f_1 = \frac{\lambda}{2} g_1$ by taking the components of $Ric_g + \nabla_g^2 f = \frac{\lambda}{2} g$ in the $M_1^{n_1}$ direction. ∎

* Tracing ① gives

$$R + \Delta f = \frac{n\lambda}{2}.$$

Divergence of ① gives

$$\nabla^i R_{ij} + \nabla^i \nabla_i \nabla_j f = 0$$

$$\Rightarrow \quad \frac{1}{2}\nabla_j R + \nabla^i \nabla_j \nabla_i f = 0 \Rightarrow \frac{1}{2}\nabla_j R + \nabla_j \nabla^i \nabla_i f - R^i{}_{jim}\nabla^m f = 0$$

$$\Rightarrow \quad \frac{1}{2}\nabla_j R + \nabla_j (\Delta f) + R_{jm}\nabla^m f = 0$$

$$\Rightarrow \quad \frac{1}{2}\nabla_j R + \nabla_j\left(\frac{n\lambda}{2} - R\right) + R_{jm}\nabla^m f = 0$$

$$\Rightarrow \quad \boxed{2R_{jm}\nabla^m f = \nabla_j R \quad \text{or} \quad \nabla R = 2Ric(\nabla f)}$$

$$\Rightarrow \quad \boxed{\langle \nabla f, \nabla R \rangle = 2Ric(\nabla f, \nabla f)}$$

Also, from $2R_{jm}\nabla^m f = \nabla_j R$, we get

$$2\left(\frac{\lambda}{2} g_{jm} - \nabla_j \nabla_m f\right)\nabla^m f = \nabla_j R$$

$$\Rightarrow \quad \lambda \nabla_j f - 2\nabla_j \nabla_m f \nabla^m f = \nabla_j R$$

$$\Rightarrow \quad \nabla_j\left(R + |\nabla f|^2 - \lambda f\right) = 0$$

$\therefore$ we get the following fundamental and important identity obtained by Hamilton

$$\boxed{R + |\nabla f|^2 - \lambda f = \text{Constant}} \quad \text{——} \quad ②$$

If $\lambda = \pm 1$ then adding a constant to the potential function f we may assume $C = 0$ $= 0$

$$R + |\nabla f|^2 = \lambda f$$ (for shrinking or expanding GRS)

for $\lambda = 0$ by scaling the metric we may assume $C = 1$ so

$$R + |\nabla f|^2 = 1$$ (for steadies).

Def^n (Bakry–Emery Ricci tensor). The Bakry–Emery Ricci tensor or the f-Ricci tensor is defined as

$$Ric_f = Ric + \nabla^2 f$$

So the GRS becomes $Ric_f = \frac{\lambda}{2} g$.

Def^n :- The f-Laplacian is given by

$$\Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle.$$

This is the natural Laplacian for computation regarding GRS.

note that if $\alpha, \beta : M^n \to \mathbb{R}$ then

$$\int_M \alpha \Delta_f \beta e^{-f} d\mu = \int \alpha (\Delta\beta - \nabla\beta \nabla f) e^{-f} d\mu$$

$$= \int -\nabla^i \beta \nabla_i (\alpha e^{-f}) d\mu - \int \alpha \nabla\beta \nabla f e^{-f} d\mu$$

$$= \int -\nabla^i \beta \nabla_i \alpha \, e^{-f} d\mu + \int \alpha \nabla^i \beta \cancel{\nabla_i f} e^{-f} d\mu$$

$$- \int \alpha \nabla \beta \cancel{\nabla f} e^{-f} d\mu$$

$$= - \int - \nabla \beta \nabla \alpha \, e^{-f} d\mu = \int \beta \Delta_f \alpha \, e^{-f} d\mu.$$

$\therefore$ Δ_f is formally self-adjoint on $L^2(e^{-f} d\mu)$.

We also have the following 3 cases:-

1) If (M^n, g, λ, f) is a shrinking GRS, then

$$R + |\nabla f|^2 = f \implies R \leq f.$$

Also, $\Delta_f f = \Delta f - \nabla f \cdot \nabla f = \Delta f - |\nabla f|^2 = \frac{n}{2} - f$

$\therefore$ $f - \frac{n}{2}$ is an eigenfunction of $-\Delta_f$ w/ eigenvalue 1.

2) For non-Ricci-flat steady GRS, $R + |\nabla f|^2 = 1$

$$\implies R \leq 1 \text{ and } \Delta_f f = -1.$$

3) For expanding GRS, $R + |\nabla f|^2 = -f \implies R \leq -f$

and $\Delta_f f = f - \frac{n}{2}$.

§ Sharp lower bounds for the scalar curvature

Thm [B.L. Chen. 09, Z.-H. Zhang, 09 Pigola–Rimoldi–Setti 2011]

(Sharp R bounds for GRS) If (M^n, g, X, λ) is a complete Ricci soliton

then a) $R \geq 0$ if $\lambda \geq 0$.

b) $R \geq \frac{\lambda n}{2}$ if $\lambda < 0$.

moreover, if equality holds at any point $\in M^n$ then (M^n, g) is Einstein. If $\lambda > 0$ and the shrinker is gradient, i.e., $X = \nabla f$. then if $R = 0$ at some point then (M^n, g, f) is a Gaussian shrinker.

Cor:- (potential function estimates) for (M^n, g, f, λ) GRS w/ $p \in M^n$.

1) on a shrinking GRS w/ $\lambda = 1$

$$|\nabla f|^2 \leq f, \quad R \leq f, \quad \Delta f \leq \frac{n}{2} \quad \text{and} \quad \sqrt{f}(x) \leq \sqrt{f}(p) + \frac{1}{2} d(x, p)$$

where $d(x,p) =$ Riemannian distance b/w x and p.

If $O \in M^n$ is a point where f attains its minima (such a point always exist by a result to be seen later) then $0 \leq R(O) = f(O) \leq \frac{n}{2}$

and $f(x) \leq \frac{1}{4}\left(d(x, O) + \sqrt{2n}\right)^2$.

2) On a steady GRS $(\lambda = 0)$

$$|\nabla f|^2 \le 1, \quad R \le 1, \quad \Delta f \le 0 \quad \text{and} \quad |f(x) - f(p)| \le d(x,p).$$

3) On an expanding GRS $(\lambda = -1)$

$$|\nabla f|^2 \le \frac{n}{2} - f, \quad \Delta f \le 0 \quad \text{and} \quad \sqrt{\frac{n}{2} - f(x)} \le \sqrt{\frac{n}{2} - f(p)} + \frac{1}{2} d(x,p).$$

$$f \le \frac{n}{2}.$$

Proof :- $\because$ $R + \Delta f = \dfrac{n\lambda}{2} \Rightarrow$ if $\lambda = 1$, $R \ge 0$

$\Rightarrow \Delta f \le \dfrac{n}{2}$

• if $\lambda = 0$, $R \ge 0 \Rightarrow \Delta f \le 0$

• if $\lambda = -1$, $R > 0 \Rightarrow \Delta f \le 0$.

Δf estimate

now, $R + |\nabla f|^2 = \lambda f \Rightarrow$ for $\lambda = 1$, $R \le f$,

and $R + |\nabla f|^2 = 1$ for $\lambda = 0$, $R \le 1$

Similarly for $\lambda = 1$, $R \ge 0$, $|\nabla f|^2 \le f$.

$\lambda = 0$, $R \ge 0$, $|\nabla f|^2 \le 1$.

$\lambda = -1$, $R \ge \dfrac{\lambda n}{2}$, we get $|\nabla f|^2 = -f - R$

$$\le -f + \frac{n}{2}.$$

$|\nabla f|$ estimate

now let's look at the shrinking soliton case:- $R \geq 0 \Rightarrow 0 \leq |\nabla f|^2 \leq f$

or $|\nabla \sqrt{f}| \leq \frac{1}{2}$ whenever $f > 0 \Rightarrow \sqrt{f}$ is Lipschitz (bounded derivative $\Rightarrow$ Lipschitz). and $\therefore$

for $f \leq 0$, the statement is obviously true.

$$\left| \sqrt{f(x)} - \sqrt{f(p)} \right| \leq \frac{1}{2} d(x,p)$$

$$\Rightarrow \quad \sqrt{f(x)} \leq \frac{1}{2} d(x,p) + \sqrt{f(p)}.$$

now suppose f attains its minima at $o \in M^n$. Then

$$f(o) - R(o) = |\nabla f|^2(o) = 0$$

$$\Rightarrow \quad 0 \leq \Delta_f f(o) = \frac{n}{2} - f(o) \Rightarrow 0 \leq f(o) \leq \frac{n}{2}$$

$\therefore$ above inequality w/ $p = o$ gives

$$\sqrt{f(x)} \leq \frac{1}{2} d(x,o) + \sqrt{\frac{n}{2}}$$

$$\Rightarrow \quad f(x) \leq \frac{1}{4} \left(d(x,o) + \sqrt{2n} \right)^2.$$

for steady GRS, $|\nabla f|^2 \leq 1 \Rightarrow$ f is Lipschitz $\Rightarrow |f(x) - f(p)| \leq d(x,p)$.

Similarly for the expanding case.

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